February 1, 2008

# On Uniqueness of the Jump Process in Event Enhanced Quantum Theory

A. Jadczyk[1], G. Kondrat and R. Olkiewicz

*Institute of Theoretical Physics, University of Wrocław*
*Pl. Maxa Borna 9, PL-50 204 Wrocław*

**Abstract**

We prove that, contrary to the standard quantum theory of continuous observation, in the formalism of Event Enhanced Quantum Theory the stochastic process generating sample histories of pairs (observed quantum system,observing classical apparatus)is unique. This result gives a rigorous basis to the previous heuristic argument of Blanchard and Jadczyk.

[1] e-mail: ajad@ift.uni.wroc.pl

# 1 Introduction

Effective time evolution of a quantum system is usually described by a dynamical semigroup: a semigroup of completely positive, unit preserving, transformations acting on the algebra of observables of the system. A general form of generator of a norm–continuous semigroup was published in 1976 independently by Gorini, Kossakowski and Sudarshan [1] (for matrix algebras) on one hand, and by Lindblad [2] (for more general, norm-continuous case) on the other. It is usually referred to as the Lindblad form; it reads:

$$\dot{A} = i[H, A] + \sum_\alpha V_\alpha^* A V_\alpha - \frac{1}{2}\{\Lambda, A\} \tag{1}$$

where $H = H^*$ is the Hamiltonian, $\{\,,\,\}$ stands for anticommutator, and

$$\Lambda = \sum_\alpha V_\alpha^* V_\alpha. \tag{2}$$

In a contrast to a pure unitary evolution that describes closed systems and which is time-reversible, the second, dissipative part of the generator makes the evolution of an open system irreversible. This irreversibility is not evident from the very form of the equation, it is connected with the positivity property of the evolution. Formally we can often solve the evolution equation backward in time, but positivity of the reversed evolution will be lost.

We can also look at the dual time evolution of states rather than of observables. For states, described by density matrices, we get:

$$\dot{\rho} = -i[H, A] + \sum_\alpha V_\alpha \rho V_\alpha^* - \frac{1}{2}\{\rho, \Lambda\}, \tag{3}$$

where the duality is defined by Tr $(\dot{A}\rho) =$ Tr $(A\dot{\rho})$.

Here again only propagation forward in time is possible, when we try to propagate backward, then we will have to deal with negative probabilities. This irreversibility is reflected in the fact that pure states evolve into mixed states. How do mixed states arise? In quantum theory, similarly as in the classical theory, they arise when we go from individual description to ensemble description, from maximal available information to partial information.

Or simply, they arise by mixing of pure states. Pure states are represented by one dimensional projection operators $P$. If $d\mu(P)$ is a probabilistic measure on pure states, then the density matrix $\rho$ defined by $\rho = \int P d\mu(P)$ is a mixed state, unless $d\mu(P)$ is a Dirac measure. Contrary to the classical theory, however, in quantum theory decomposition of a mixed state into pure ones is non–unique. So, for instance, the identity operator can be decomposed into *any* complete orthonormal basis: $I = \sum_i |i >< i|$, thus in indenumerably many ways. This mysterious and annoying non-uniqueness of decomposition into pure states in quantum theory can be simply taken as an unavoidable price for our progress from classical to quantum, as a fact of life. And so it was. Yet it started to cause problems in quantum measurement theory.

The first attempt to give a precise mathematical formulation of quantum measurement theory must be ascribed to John von Neumann. In his monograph [3] he introduced two kind of evolutions: a continuous, unitary evolution $U$ of an 'unobserved' system, and discontinuous 'projections' that accompany 'observations' or 'measurements'. His projection postulate, later reformulated by Lüders for mixed states, is expressed as follows:

'if we measure a property $E$ of the quantum system, and if we do not make any filtering which depends on the result, then as the result of this measurement the system which was previously described by a density matrix $\rho$ switches to the new state described by the density matrix $E\rho E/\mathrm{Tr}\ E\rho E$.'

A whole generation of physicists was brainwashed by this apparently precise formulation. Few dared to ask: who are 'we' in the phrase 'if we measure' [4], what is 'measurement' [5, 6], at which particular instant of time the reduction takes place? How long does it take [7], if ever [8], to reduce? Can it be observed? Can it be verified experimentally [9, 10, 11]? Nobody could satisfactorily answer these questions. And so it was taken for granted that quantum theory can not really be understood in physical terms, that it is a peculiar mixture of objective and subjective. That it is about 'observations', and so it makes little or no sense without 'observers', and without 'mind'. There were many that started to believe that it is the sign of new age and the sign of progress. Few opponents did not believe completeness of a physical

theory that could not even define what constitutes 'observation' [5, 6]– but they could not change the overall feeling of satisfaction with successes of the quantum theory.

This situation started to change rapidly when technological progress make it possible to make prolonged experiments with individual quantum systems. The standard 'interpretation' did not suffice. Experimenters were seeing with their own eyes not the 'averages' but individual sample histories. In particular, experiments in Quantum Optics allowed one to almost 'see' the quantum jumps. In 1988 J.R. Cook [12] discussed photon counting statistics in fluorescence experiments and revived the question 'what are quantum jumps?'. Another reason to pay more attention to the notion of quantum jumps came from the several groups of physicists working on effective numerical solutions of quantum optics master equations. The works of Carmichael [13], Dalibard, Castin and Mølmer [14, 15], Dum, Zoller and Ritsch [16], Gardiner, Parkins and Zoller [17], developed the method of Quantum Trajectories, or Quantum Monte Carlo (QMC) algorithm for simulating solutions of master equations. It was soon realized (cf. e.g. [18, 19, 20, 21, 22]) that the same master equations can be simulated either by Quantum Monte Carlo method based on quantum jumps, or by a continuous quantum state diffusion. Wiseman and Milburn [23] discussed the question of how different experimental detection schemes relate to continuous diffusions or to discontinuous jump simulations. The two approaches were recently put into comparison also by Garraway and Knight [24]. There are at present two schools of simulations. Gisin et al. [25] tried to reconcile the two arguing that '*the quantum jumps can be clearly seen*' also in the quantum state diffusion plots. On the other hand already in 1986 Diosi [26] proposed a pure state, piecewise deterministic process that reproduces a given master equation. In spite of the title of his paper that suggests uniqueness of his scheme, his process although mathematically canonical for a given master equation - it is not unique.
This problem of non-uniqueness is especially important in theories of gravity-induced spontaneous localization (see [27], also [28, 29] and references therein) and in the recent attempts to merge mind-brain science with quantum theory [30, 31, 32], where quantum collaps plays an important role.

In the next section we shall see how the situation changes completely with the new approach to quantum measurement developed by Ph. Blanchard and one of us (see [33] and references there)[2]. In Sec. 2 we will sketch the main idea of the new approach. We will also indicate infinitesimal proof of uniqueness of the stochastic process that reproduces master equation for the total system, i.e. quantum system+classical apparatus. In Sec. 3 we give concrete examples of non-unicity when only a pure quantum system is involved - as it is typical in quantum optics. In Sec. 4. we will give a rigorous, global proof of unicity of the process, when classical apparatus is coupled in an appropriate way to the quantum system. Conclusions will be given in Sec. 5. There we also comment upon the most natural question: we all know that every apparatus consists of atoms - then how can it be classical?

# 2 The formalism

Let us sketch the mathematical framework of the "event-enhanced quantum theory". Details can be found in [33]. To describe events, one needs a classical system $C$, then possible events are identified with changes of a (pure) state of $C$. One can think of events as 'clicks' of a particle counter, changes of the pointer position, changing readings on an apparatus LCD display. The concept of an event is of course an idealization - like all concepts in a physical theory. Let us consider the simplest situation corresponding to a finite set of possible events. The space of pure states of $C$, denoted by $\mathcal{S}_c$, has $m$ states, labeled by $\alpha = 1, \ldots, m$. Statistical states of $C$ are probability measures on $\mathcal{S}_c$ – in our case just sequences $p_\alpha \geq 0, \sum_\alpha p_\alpha = 1$.

The algebra of observables of $C$ is the algebra $\mathcal{A}_c$ of complex functions on $\mathcal{S}_c$ – in our case just sequences $f_\alpha, \alpha = 1, \ldots, m$ of complex numbers.

We use Hilbert space language even for the description of the classical system. Thus we introduce an $m$-dimensional Hilbert space $\mathcal{H}_c$ with a fixed basis, and we realize $\mathcal{A}_c$ as the algebra of diagonal matrices $F = \mathrm{diag}(f_1, \ldots, f_m)$.

[2] Complete, actual bibliography of the Quantum Future Project is always available under URL: http://www.ift.uni.wroc.pl~˜ajad/qf-pub.htm

Statistical states of $C$ are then diagonal density matrices $\mathrm{diag}(p_1, \ldots, p_m)$, and pure states of $C$ are vectors of the fixed basis of $\mathcal{H}_c$.
Events are ordered pairs of pure states $\alpha \to \beta$, $\alpha \neq \beta$. Each event can thus be represented by an $m \times m$ matrix with 1 at the $(\alpha, \beta)$ entry, zero otherwise. There are $m^2 - m$ possible events.

We now come to the quantum system.
Let $Q$ be the quantum system whose bounded observables are from the algebra $\mathcal{A}_q$ of bounded operators on a Hilbert space $\mathcal{H}_q$. In this paper we will assume $\mathcal{H}_q$ to be *finite dimensional*. Pure states of $Q$ are unit vectors in $\mathcal{H}_q$; proportional vectors describe the same quantum state. Statistical states of $Q$ are given by non–negative density matrices $\hat{\rho}$, with $\mathrm{Tr}\ (\hat{\rho}) = 1$.

Let us now consider the total system $T = Q \times C$. For the algebra $\mathcal{A}_t$ of observables of $T$ we take the tensor product of algebras of observables of $Q$ and $C$: $\mathcal{A}_t = \mathcal{A}_q \otimes \mathcal{A}_c$. It acts on the tensor product $\mathcal{H}_q \otimes \mathcal{H}_c = \oplus_{\alpha=1}^m \mathcal{H}_\alpha$, where $\mathcal{H}_\alpha \approx \mathcal{H}_q$. Thus $\mathcal{A}_t$ can be thought of as algebra of *diagonal* $m \times m$ matrices $A = (a_{\alpha\beta})$, whose entries are quantum operators: $a_{\alpha\alpha} \in \mathcal{A}_q$, $a_{\alpha\beta} = 0$ for $\alpha \neq \beta$.
Statistical states of $Q \times C$ are given by $m \times m$ diagonal matrices $\rho = \mathrm{diag}(\rho_1, \ldots, \rho_m)$ whose entries are positive operators on $\mathcal{H}_q$, with the normalization $\mathrm{Tr}\ (\rho) = \sum_\alpha \mathrm{Tr}\ (\rho_\alpha) = 1$. Duality between observables and states is provided by the expectation value $< A >_\rho = \sum_\alpha \mathrm{Tr}\ (A_\alpha \rho_\alpha)$.

We will now generalize slightly our framework. Indeed, there is no need for the quantum Hilbert spaces $\mathcal{H}_\alpha$, corresponding to different states of the classical system, to coincide. We will allow them to be different in the rest of this paper. We denote $n_\alpha = dim(\mathcal{H}\alpha)$.

We consider now dynamics. It is normal in quantum theory that classical parameters enters quantum Hamiltonian. Thus we assume that quantum dynamics, when no information is transferred from $Q$ to $C$, is described by Hamiltonians $H_\alpha : \mathcal{H}_\alpha \longrightarrow \mathcal{H}_\alpha$, that may depend on the actual state of $C$ (as indicated by the index $\alpha$). We will use matrix notation and write $H = \mathrm{diag}(H_\alpha)$. Now take the classical system. It is discrete here. Thus it can not have continuous time dynamics of its own.

As in [33] the *coupling* of $Q$ to $C$ is specified by a matrix $V = (g_{\alpha\beta})$, where $g_{\alpha\beta}$ are linear operators: $g_{\alpha\beta} : \mathcal{H}_\beta \longrightarrow \mathcal{H}_\alpha$. We put $g_{\alpha\alpha} = 0$. This condition expresses the simple fact: we do not need dissipation without receiving information (i.e without an event). To transfer information from $Q$ to $C$ we need a non–Hamiltonian term which provides a completely positive (CP) coupling. As in [33] we consider couplings for which the evolution equation for observables and for states is given by the Lindblad form:

$$\dot{A}_\alpha = i[H_\alpha, A_\alpha] + \sum_\beta g^\star_{\beta\alpha} A_\beta g_{\beta\alpha} - \frac{1}{2}\{\Lambda_\alpha, A_\alpha\}, \tag{4}$$

or equivalently:

$$\dot{\rho}_\alpha = -i[H_\alpha, \rho_\alpha] + \sum_\beta g_{\alpha\beta} \rho_\beta g^\star_{\alpha\beta} - \frac{1}{2}\{\Lambda_\alpha, \rho_\alpha\}, \tag{5}$$

where

$$\Lambda_\alpha = \sum_\beta g^\star_{\beta\alpha} g_{\beta\alpha}. \tag{6}$$

The above equations describe statistical behavior of ensembles. Individual sample histories are described by a Markov process with values in pure states of the total system. In [33] this process was argued to be infinitesimally unique. For the sake of completeness we repeat here the arguments. First, we use Eq. (5) to compute $\rho_\alpha(dt)$ when the initial state $\rho_\alpha(0)$ is pure:

$$\rho_\alpha(0) = \delta_{\alpha\alpha_0} |\psi_0 >< \psi_0|. \tag{7}$$

In the equations below we will discard terms that are higher than linear order in $dt$. For $\alpha = \alpha_0$ we obtain:

$$\begin{aligned} \rho_{\alpha_0}(dt) = \quad |\psi_0 >< \psi_0| \quad & -i[H_{\alpha_0}, |\psi_0 >< \psi_0|]\, dt - \\ & -\tfrac{1}{2}\{\Lambda_{\alpha_0}, |\psi_0 >< \psi_0|\}\, dt, \end{aligned} \tag{8}$$

while for $\alpha \neq \alpha_0$

$$\rho_{\alpha_0}(dt) = g_{\alpha\alpha_0} |\psi_0 >< \psi_0| g^\star_{\alpha\alpha_0}\, dt \tag{9}$$

The term for $\alpha = \alpha_0$ can be written as

$$\rho_{\alpha_0}(dt) = p_{\alpha_0} |\psi_{\alpha_0} >< \psi\alpha_0|, \tag{10}$$

where

$$\psi_{\alpha_0} = \frac{\exp\left(-iH_{\alpha_0}dt - \frac{1}{2}\Lambda_{\alpha_0}dt\right)\psi_0}{\|\exp\left(-iH_{\alpha_0}dt - \frac{1}{2}\Lambda_{\alpha_0}dt\right)\psi_0\|}, \tag{11}$$

and

$$p_{\alpha_0} = 1 - \lambda(\psi_0, \alpha_0)dt. \tag{12}$$

The term with $\alpha \neq \alpha_0$ can be written as:

$$\rho_\alpha(dt) = p_\alpha \left|\psi_\alpha >< \psi_\alpha\right|, \tag{13}$$

where

$$p_\alpha = \|g_{\alpha\alpha_0}\psi_0\|^2 dt, \tag{14}$$

and

$$\psi_\alpha = \frac{g_{\alpha\alpha_0}\psi_0}{\|g_{\alpha\alpha_0}\psi_0\|} \tag{15}$$

This representation is unique and it defines the infinitesimal version of a piecewise deterministic Markov process.

# 3 Non-uniqueness in the pure quantum case

In this section we will show on simple examples the nature of non-uniqueness in the pure quantum case.

For simplicity let us consider a two state quantum system whose algebra of observables is equal to $M_{2\times 2}$. Let $T_t$ be a dynamical semigroup with a generator $L$ given by

$$L(\rho) \; = \; a\rho a^* \; - \; \frac{1}{2}\{a^* a \, , \, \rho\},$$

where $a \in M_{2\times 2}$.

## 3.1 Pure diffusion process

First let us show that the time evolution determined by $L$ can be described by a diffusion process with values in $\mathbf{C}P^1$ [34].

Let a two component complex valued process $\psi_t = (\psi_t^1, \psi_t^2)'$ (prime denotes the transposition) be given by the following stochastic differential equation:

$$d\psi_t^i = f_i(\psi_t)dB_t + g_i(\psi_t)dt, \;\; i = 1, 2,$$

where $B_t$ is a one-dimensional real Brownian motion and

$$g_i(\psi_t) = \sum_j (< a^* >_t a_{ij} - \frac{1}{2}(a^* a)_{ij})\psi_t^j - \frac{1}{2} < a^* >_t < a >_t \psi_t^i$$

$$f_i(\psi_t) = \sum_j a_{ij}\psi_t^j - < a >_t \psi_t^i$$

$$< a >_t = \frac{< \psi_t|a|\psi_t >}{< \psi_t|\psi_t >}, \quad < a^* >_t = \frac{\psi_t|a^*|\psi_t >}{< \psi_t|\psi_t >}$$

Moreover let us choose an initial condition $\psi_0 = (z_0^1, z_0^2)'$ such that $|z_0^1|^2 + |z_0^2|^2 = 1$. Because $f_i$ and $g_i$ are continuously differentiable (in the real sense) on $\mathbf{C}^2 \setminus \{0\}$ so there exists a local solution with a random explosion time T (see for example [35]). But

$$d|\psi_t^i|^2 = \psi_t^i d\bar{\psi}_t^i + \bar{\psi}_t^i d\psi_t^i + d[\psi_t^i, \bar{\psi}_t^i]_t,$$

where $[\psi_t^i, \bar{\psi}_t^i]_t$ is the quadratic covariation of $\psi_t^i$ and $\bar{\psi}_t^i$. Thus

$$d[\psi_t^i, \bar{\psi}_t^i]_t = |f_i(\psi_t)|^2 dt,$$

and so

$$d\|\psi_t\|^2 = \sum_i (\psi_t^i d\bar{\psi}_t^i + \bar{\psi}_t^i d\psi_t^i) + \|f(\psi_t)\|^2 dt = 0.$$

It implies that $T = \infty$ with probability one and so our process is a diffusion with values in a sphere $S^3$. Let us define a process $P_t$ with values in one dimensional projectors by

$$P_t = |\psi_t >< \psi_t| = \sum_{i,j} \psi_t^i \bar{\psi}_t^j e_{ij},$$

where $e_{ij}$ form the standard basis in $M_{2\times 2}$. Then, using the equation

$$d(\psi_t^i \bar{\psi}_t^j) = (\bar{f}_j \psi_t^i + f_i \bar{\psi}_t^j)dB_t + (\bar{g}_j \psi_t^i + g_i \bar{\psi}_t^j + f_i \bar{f}_j)dt$$

we obtain that

$$dP_t \;=\; [(a- < a >_t)P_t \;+\; P_t(a^*- < a^* >_t)]dB_t \;-\; \frac{1}{2}\{a^*a,\, P_t\}dt \;+\; aP_ta^*dt.$$

Since $B_t$ is a martingale then after taking the average we get

$$dE[P_t] \;=\; aE[P_t]a^*dt \;-\; \frac{1}{2}\{a^*a,\, E[P_t]\}dt$$

Let us define a density matrix $\rho_t = \; E[P_t]$. Then

$$\dot{\rho}_t \;=\; a\rho_t a^* \;-\; \frac{1}{2}\{a^*a,\, \rho_t\},$$

and so the average of the diffusion gives the quantum dynamical evolution.

Finally, we show that $\rho_t = \int P(t,\, x_0,\, dy)P_y$, where $P_y = |y >< y|$, $x_0 = |\psi_0 >< \psi_0|$ and $P(t,\, x,\, dy)$ is the transition probability of the described diffusion. By the definition, $P(t,\, x_0,\, \Gamma)$ is the distribution of the random variable $P_t^{x_0}$ such that $P_0^{x_0} \;=\; x_0$. It implies that for every bounded and measurable function $f$ defined on $\mathbf{C}P^1$ we have

$$E[f(P_t^{x_0})] \;=\; \int f(y)P(t,\, x_0,\, dy).$$

Let us consider a function given by $f(y) = Tr(AP_y)$, where $A \in M_{2\times 2}$. Then

$$\int Tr(AP_y)P(t,\, x_0,\, dy) \;=\; E[Tr(AP_t^{x_0})] \;=\; Tr(A\rho_t).$$

So $Tr(A \int P(t,\, x_0,\, dy)P_y)) = Tr(A\rho_t)$ for every $A$ and thus $\rho_t = \int P(t,\, x_0,\, dy)P_y$ with $\rho_0 = \; x_0$.

## 3.2 Piecewise deterministic solution

On the other hand it is possible to associate with the same quantum dynamics a piecewise deterministic process, as in the method of quantum trajectories [13]. Now the situation is more complicated, because, in general, we can not replace the Brownian motion by the Poisson process. We have to solve a stochastic differential equation for an unknown process $(\tilde{N}_t,\, \psi_t)$.

$$d\psi_t^i \;=\; f_i(\psi_{t^-})d\tilde{N}_t \;+\; g_i(\psi_t)dt,$$

where $f_i$ and $g_i$ are prescribed functions, together with the following constrain: $\tilde{N}_t$ is a semimartingale such that

a) $[\tilde{N}, \tilde{N}]_t = \tilde{N}_t$, $\tilde{N}_0 = 0$, $E[\tilde{N}_t] < \infty$ for all $t \geq 0$,

b) for a given nonnegative function $\lambda : \mathbf{C}^2 \to \mathbf{R}$ the process $M_t := \tilde{N}_t - \int_0^t \lambda(\psi_s)ds$ is a martingale.

It is clear that $M_t$ will be a purely discontinuous martingale. A continuous, increasing and with paths of finite variation on compacts process: $\int_0^t \lambda(\psi_t)ds$ is called the compensator of $\tilde{N}_t$. In our case due to assumption a) it is also the conditional quadratic variation of $\tilde{N}_t$ [35]. The functional $\lambda(\psi_t)$ is called the stochastic intensity and plays the role of the intensity of jumps. Let us recall that for the (homogeneous) Poisson process $N_t - \int_0^t \lambda ds = N_t - \lambda t$ is a martingale. From the assumption a) above we obtain that $\tilde{N}_t$ is quadratic pure jump, its continuous part is equal zero and $\triangle \tilde{N}_s = (\triangle \tilde{N}_s)^2$, where $\triangle \tilde{N}_s = \tilde{N}_s - \tilde{N}_{s^-}$ so it is a point process. Let us emphasize that in general it is not an inhomogeneous Poisson process since its compensator would be a deterministic function equal to $E[\tilde{N}_t]$ [36]. So it will be the case only when the stochastic intensity is a deterministic function depending on $t$.
Moreover $[\tilde{N}, t]_t = 0$ as $\tilde{N}_t$ is of finite variation on compacts. It implies the following symbolic rules

$$(d\tilde{N})^2 = d\tilde{N}, \quad d\tilde{N}dt = dtd\tilde{N} = 0.$$

From assumption b) we get $dM_t = d\tilde{N}_t - \lambda(\psi_t)dt$. Let $\mathcal{F}_t$ be a $\sigma$-algebra of all events up to time $t$. Because $M_t$ is a martingale, so $E[dM_t|\mathcal{F}_t] = 0$ what implies:

$$E[d\tilde{N}_t|\mathcal{F}_t] = \lambda(\psi_t)dt$$

see [38].

Till now the operator $a \in M_{2\times 2}$ was arbitrary. A particular simple case is if we take

$$a^* = a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

. Then $L(\rho_t) \;=\; a\rho_t a \;-\; \rho_t$ and so the intensity

$$\lambda(\psi_t) \;=\; < a^* a >_t = \; \frac{< \psi_t | a^* a | \psi_t >}{< \psi_t | \psi_t >} \;=\; 1$$

what implies that $\tilde{N}_t \;=\; N_t$. Because there is no deterministic evolution (we do not have the Hamiltonian part and the jump rate is constant) so in this case we can put $g_1 \;=\; g_2 \;=\; 0$ and $f_1(\psi_t) \;=\; \psi_t^2 - \psi_t^1,\; f_2(\psi_t) \;=\; \psi_t^1 - \psi_t^2$ as the probability of a particular jump depends on the difference between $\psi_t^1$ and $\psi_t^2$. Thus we arrive at

$$d\psi_t^i \;=\; f_i(\psi_{t^-}) dN_t.$$

Using the identity $d[\psi^i,\, \psi^j]_t \;=\; f_i \bar{f}_j dN_t$ we get that $d\|\psi\|^2 \;=\; 0$ and $dP_t \;=\; (aP_t a - P_t)dN_t$. Taking the average we obtain $\dot{\rho}_t = a\rho_t a - \rho_t$, since $N_t - \lambda t$ is a martingale. The above stochastic differential equation admits the following solution

$$\begin{aligned} \psi_t^1 \;&=\; z_0^1 \frac{1 + (-1)^{N_t}}{2} \;+\; z_0^2 \frac{1 - (-1)^{N_t}}{2} \\ \psi_t^2 \;&=\; z_0^1 \frac{1 - (-1)^{N_t}}{2} \;+\; z_0^2 \frac{1 + (-1)^{N_t}}{2} \end{aligned}$$

It implies that

$$P_t \;=\; x_0 \frac{1 + (-1)^{N_t}}{2} \;+\; y_0 \frac{1 - (-1)^{N_t}}{2},$$

where $x_0 \;=\; |\psi_0 >< \psi_0|$ and $y_0 \;=\; |\phi_0 >< \phi_0|, \quad \phi_0 \;=\; (a + a^*)\psi_0 \;=\; (z_0^2,\, z_0^1)'$.

If we take

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

as it is usual in quantum optics problems, then we have

$$\lambda(\psi_t) \;=\; \frac{|\psi_t^2|^2}{\|\psi_t\|^2}.$$

So we need a point process whose rate function is random and the situation is slightly more complicated. We have to use the more general method described at the beginning of this paragraph.

Let us start with calculating functions $g_i$, which are responsible for the deterministic flow. They are obtained by taking the derivative of

$$\psi_s \ = \ \frac{exp(-\frac{1}{2}sa^*a)\psi_t}{\|exp(-\frac{1}{2}sa^*a)\psi_t\|}\|\psi_t\|$$

with respect to $s$ and in the instant $s \ = \ 0$. So we get

$$g(\psi_t) \ = \frac{1}{2}(-a^*a + < a^*a >_t)\psi_t.$$

It can be checked that the only functions $f_i$ which lead to the Lindblad equation are of the following type:

$$f_1(\psi_t) \ = \ -\psi_t^1 \ + \sqrt{< \psi_t|\psi_t >}\, e^{ih(\psi_t)}, \quad f_2(\psi_t) \ = \ -\psi_t^2,$$

where $h : \mathbf{C}^2 \to \mathbf{R}$ is an arbitrary Lipschitz function. Let us point out that if we put $e^{ih} \ = \ \psi_t^2/|\psi_t^2|$ then we can write $f$ in a compact form

$$f(\psi_t) \ = \ (\frac{a}{\sqrt{< a^*a >_t}} \ - \ \mathbf{1})\psi_t$$

see [38], but it needs a careful interpretation because zero can appear in the denominator. Again by simple calculations we get that $d\|\psi_t\|^2 \ = \ 0$ and

$$dP_t \ = \begin{pmatrix} |\psi_t^2|^2 & -\psi_t^1\bar{\psi}_t^2 \\ -\bar{\psi}_t^1\psi_t^2 & -|\psi_t^2|^2 \end{pmatrix}_- d\tilde{N}_t$$

$$+\frac{1}{2 < \psi_t|\psi_t >}\begin{pmatrix} 2|\psi_t^1|^2|\psi_t^2|^2 & \psi_t^1\bar{\psi}_t^2(|\psi_t^2|^2 - |\psi_t^1|^2) \\ \bar{\psi}_t^1\psi_t^2(|\psi_t^2|^2 - |\psi_t^1|^2) & -2|\psi_t^1|^2|\psi_t^2|^2 \end{pmatrix} dt.$$

But $d\tilde{N}_t \ = \ dM_t \ + \ \lambda(\psi_t)dt$ so after averaging we get the quantum evolution equation for $\rho_t \ = \ E[P_t]$.

# 4 Global existence and uniqueness

After analyzing a typical example of non uniqueness in the pure quantum case, here we will return to the general scheme as described in Section 2. Let $T_t$ be a norm-continuous dynamical semigroup on states of the total

algebra $\mathcal{A}_T$ corresponding to eq. (5). We extend $T_t$ by linearity to the whole predual space $\mathcal{A}_{T*}$, which is equal to $\mathcal{A}_T$, because the total algebra is finite dimensional. Let $E$ denote a space of all one-dimensional projectors in $\mathcal{A}_T$. Because $\mathcal{A}_T = \oplus_{\alpha=1}^{\alpha=m} M(n_\alpha \times n_\alpha)$ we obtain that $E = \dot{\cup}_\alpha \mathbf{C}P_\alpha$ and so $E$ is a disjoint sum of compact differentiable manifolds (complex projective spaces in $\mathcal{H}_\alpha$). We would like to associate with $T_t$ a homogeneous Markov – Feller process with values in $E$ such that for every $x \in E$

$$T_t(P_x) = \int_E P(t,x,dy) P_y, \tag{16}$$

where $P(t,x,dy)$ is the transition probability function for the process $\xi_t$ and $y \to P_y$ is a map which assigns to every point $y \in E$ a one-dimensional projector $P_y$. This leads us to the following definition.

Let $\mathcal{M}(E)$ denote a Banach space of all complex, finite, Borel measures on $E$. We say that a positive and contractive semigroup $U_t : \mathcal{M}(E) \to \mathcal{M}(E)$ with a Feller transition function $P(t,x,\Gamma)$ is associated with $T_t$ iff Eq.16 is satisfied.

Let us describe this notion more precisely. Let $\pi$ be a map between two Banach spaces $\mathcal{M}(E)$ and $\mathcal{A}_{T*}$ given by

$$\pi(\mu) = \int_E \mu(dx)\, P_x$$

It is clear that $\pi$ is linear, surjective, preserves positive cones and $\|\pi\| = 1$.

**Proposition 1.** $U_t$ is associated with $T_t$ iff $ker\, \pi$ is $U_t$ – invariant and $\hat{U}_t = T_t$, where $\hat{U}_t$ is the quotient group of $U_t$ by $ker\, \pi$.

**Proof.** Let $U_t$ be associated with $T_t$. It implies that

$$\int_E P(t,x,dy)\, P_y \; = T_t(P_x)$$

thus for any $\mu_0 \in ker\, \pi$ we have

$$\int_E (U_t \mu_0)(dx)\, P_x = \int_E \int_E P(t,y,dx)\, \mu_0(dy)\, P_x =$$

$$\int_E T_t(P_y)\, \mu_0(dy) = T_t\, [\int_E \mu_0(dy)\, P_y] = 0$$

and so $U_t\mu_0 \in ker\,\pi$. Moreover $\forall \mu \in \mathcal{M}(E)$

$$\hat{U}_t\,\pi(\mu) = \pi(U_t\mu) = \int_E (U_t\mu)(dy)\,P_y =$$

$$\int_E\int_E P(t,x,dy)\mu(dx)\,P_y = T_t[\int_E \mu(dx)\,P_x] = T_t\pi(\mu).$$

Now let us assume that $\hat{U}_t = T_t$ i.e. $\forall\mu \in \mathcal{M}(E)$ we have $\hat{U}_t\pi(\mu) = T_t\pi(\mu)$. Let us take $\mu = \delta_x$. Then

$$\hat{U}_t\pi(\delta_x) = \pi(U_t\delta_x) = \int_E (U_t\delta_x)(dy)P_y =$$

$$\int_E\int_E P(t,z,dy)\delta_x(dz)P_y = \int_E P(t,x,dy)P_y$$

and $T_t\pi(\mu) = T_t(P_x)$ so $T_t(P_x) = \int_E P(t,x,dy)P_y$. $\square$

It means that to find $U_t$ is to extend the semigroup $T_t$ from $\mathcal{M}(E)/ker\,\pi$ to $\mathcal{M}(E)$ in an invariant way. It should be emphasized that, in general, such an 'extension' may not exist or, if it exists, need not be unique. We show that in our case, under mild assumptions, the existence and the uniqueness can be proved.

Let us write the evolution equation for states in the Lindblad form

$$\dot{\rho} = -i[H,\rho] + \sum_k V_k^*\rho V_k - \frac{1}{2}\{\rho, \sum_k V_k V_k^*\},$$

where $H = diag(H_1,\ldots,H_m)$, $H_\alpha = H_\alpha^* \in M(n_\alpha \times n_\alpha)$ and $V_k$ satisfy the following assumptions:

a) $(V_k)_{\alpha\alpha} = 0$ for every $k$ and $\alpha$

b) if for some $k,l,\alpha,\beta$ $(V_k)_{\alpha\beta} \neq 0$ and $(V_l)_{\alpha\beta} \neq 0$ then $k = l$[3]

Let $A$ be a densely defined linear operator on $C(E)$ with $D(A) = C^1(E)$ given by

$$(Af)(x) = \sum_{\alpha\neq\alpha_0} c_\alpha(x)f(x_\alpha) - c(x)f(x) + v(x)f,$$

[3] In general we can allow for a weaker version: $(V_k)_{\alpha\beta} \neq 0$ and $(V_l)_{\alpha\beta} \neq 0 \Rightarrow \exists c \in \mathbf{C} : (V_k)_{\alpha\beta} = c(V_l)_{\alpha\beta}$, but this simply reduces to b) above by substitution $(\tilde{V}_k)_{\alpha\beta} := \sqrt{1+|c|^2}(V_k)_{\alpha\beta}$ and $(\tilde{V}_l)_{\alpha\beta} = 0$ for $k \neq l$.

where $x \in \mathbf{C}P_{\alpha_0}$, $c_\alpha(x) = \mathrm{Tr}\ (P_x W_{\alpha_0\alpha} W^*_{\alpha_0\alpha})$, $W_{\alpha_0\alpha} = \sum_k (V_k)_{\alpha_0\alpha} \in L(H_\alpha, \mathcal{H}_{\alpha_0})$, $W^*_{\alpha_0\alpha} = \sum_k (V_k)^*_{\alpha_0\alpha} \in L(\mathcal{H}_{\alpha_0}, H_\alpha)$, $c(x) = \sum_{\alpha\neq\alpha_0} c_\alpha(x)$, $P_{x_\alpha} = \frac{W^*_{\alpha_0\alpha} P_x W_{\alpha_0\alpha}}{\mathrm{Tr}\ (P_x W_{\alpha 0\alpha} W^*_{\alpha_0\alpha})} \in \mathbf{C}P_\alpha$ and $x \to v(x)$ is a vector field on $E$ such that

$$v(x) = -i[H_{\alpha_0}, P_x] - \frac{1}{2}\{P_x, \sum_{\alpha\neq\alpha_0} W_{\alpha_0\alpha} W^*_{\alpha_0\alpha}\} + P_x \mathrm{Tr}\ (P_x \sum_{\alpha\neq\alpha_0} W_{\alpha_0\alpha} W^*_{\alpha_0\alpha})$$

It may be easily checked that $v(x) \in T_x \mathbf{C}P_\alpha = T_x E$. Because

$$g_t(P_x) = \frac{\exp[t(-iH_{\alpha_0} - \frac{1}{2}\sum_{\alpha\neq\alpha_0} W_{\alpha_0\alpha} W^*_{\alpha_0\alpha})] P_x \exp[t(iH_{\alpha_0} - \frac{1}{2}\sum_{\alpha\neq\alpha_0} W_{\alpha_0\alpha} W^*_{\alpha_0\alpha})]}{\mathrm{Tr}\ (P_x \exp[-t\sum_{\alpha\neq\alpha_0} W_{\alpha_0\alpha} W^*_{\alpha_0\alpha}])}$$

is an integral curve for $v$, so we have that $v$ is a complete vector field.

**Theorem 2.** *$A$ is a generator of a strongly continuous positive semigroup of contractions $S_t$ on $C(E)$.*

**Proof.** $A = A_1 + A_2$, where $(A_1 f)(x) = \sum_{\alpha\neq\alpha_0} c_\alpha(x)\delta_{x_\alpha} f - c(x)\delta_x f$ and $A_2 = v$. It is clear that $A_1$ is a bounded and dissipative operator. It is also a dissipation i.e. $A_1(f^2) \geq 2f A_1(f)$ for $f = \bar{f}$. Because $A_2$ generates a flow on $E$ given by $f(x) \to f(g_t(x))$, where $g_t(x)$ is the integral curve of $v$ starting at the point $x$, it follows that $A = A_1 + A_2$ is the generator of a strongly continuous semigroup of contractions (see for example [46]). Positivity follows from the Trotter product formula, since both $A_1$ and $A_2$ generates positive semigroups. $\square$

Let $P(t, x, \Gamma)$ denote the transition function of $S_t$.

**Proposition 3.** *$P(t, x, \Gamma)$ is a Feller transition function.*

**Proof.** By theorem 2.8 of [39], and by conservativeness of $P(t, x, \Gamma)$, it is enough to show that for every $x \in E$ and for any $f \in C^1(E)$ such that $f(x) = 0$, $f(y) \leq 0\ \forall y \in E$ we have $(Af)(x) \leq 0$. Because $f$ has a maximum at $x$ so $(A_2 f)(x) = 0$. Moreover, as $x \in \mathbf{C}P_{\alpha_0}$ for some $\alpha_0$ and $c_\alpha(x) \geq 0$ $\forall \alpha \neq \alpha_0$ we have

$$(A_1f)(x) = \sum_{\alpha\neq\alpha_0} c_\alpha(x)f(x_\alpha) \leq 0$$

$\square$

Now prove that that our process reproduces $T_t$.

**Theorem 4.** Let $(U_t\mu)(\Gamma) := \int_E P(t,x,\Gamma)\mu(dx)$ for $\mu \in \mathcal{M}(E)$. Then $U_t$ is associated with $T_t$.

**Proof.** At first we show that $\forall x \in E$

$$L(P_x) = [A(P)](x), \tag{17}$$

where $L$ is the generator of $T_t$, $A$ is the generator of $S_t$ and $P : x \to P_x$.

Let $x \in \mathbf{C}P_{\alpha_0}$. In $\mathcal{H} = \oplus_{\alpha=1}^m \mathcal{H}_\alpha$ let us choose any orthonormal basis $\{e_{\alpha,i_\alpha}\}_{i_\alpha=1,\dots,n_\alpha}^{\alpha=1,\dots,m}$, for which $e_{\alpha,i_\alpha} \in \mathcal{H}_\alpha$. Obviously, for any $P_x \in \mathcal{A}_{T*}$ $< e_{\alpha,i_\alpha}|L(\rho)|e_{\beta,i_\beta} >= 0$ for $\alpha \neq \beta$ and the same is true for $[A(P)](x)$. So it is enough to evaluate the $(\beta, i_\beta, j_\beta)$-th matrix elements of both sides of Eq.(17):

$$< e_{\beta,i_\beta}|[A(P)](x)|e_{\beta,j_\beta} >= \sum_{\alpha\neq\alpha_0} \mathrm{Tr}\,(P_xW_{\alpha_0\alpha}W^*_{\alpha_0\alpha})\cdot\frac{< e_{\beta,i_\beta}|W^*_{\alpha_0\alpha}P_xW_{\alpha_0\alpha}|e_{\beta,j_\beta} >}{\mathrm{Tr}\,(P_xW_{\alpha_0\alpha}W^*_{\alpha_0\alpha})} -$$

$$-\sum_{\alpha\neq\alpha_0} \mathrm{Tr}\,(P_xW_{\alpha_0\alpha}W^*_{\alpha_0\alpha}) < e_{\beta,i_\beta}|P_x|e_{\beta,j_\beta} > +$$

$$< e_{\beta,i_\beta}|(-i[H_{\alpha_0},P_x]-\frac{1}{2}\{P_x,\sum_{\alpha\neq\alpha_0} W_{\alpha_0\alpha}W^*_{\alpha_0\alpha}\}+P_x\,\mathrm{Tr}\,(P_x\sum_{\alpha\neq\alpha_0} W_{\alpha_0\alpha}W^*_{\alpha_0\alpha}))|e_{\beta,j_\beta} >=$$

$$=< e_{\beta,i_\beta}|W^*_{\alpha_0\beta}P_xW_{\alpha_0\beta}|e_{\beta,j_\beta} > \ +\delta_{\alpha_0\beta} < e_{\beta,i_\beta}|(-i[H_{\alpha_0},P_x]-$$

$$\frac{1}{2}\{P_x,\sum_{\alpha\neq\alpha_0} W_{\alpha_0\alpha}W^*_{\alpha_0\alpha}\})|e_{\beta,j_\beta} > \tag{18}$$

On the other hand the $\beta$-th component of $L(P_x)$

$$(L(P_x))_\beta = \sum_k (V_k)^*_{\alpha_0\beta}P_x(V_k)_{\alpha_0\beta}+\delta_{\alpha_0\beta}-(i[H_{\alpha_0},P_x]+\frac{1}{2}\{P_x,\sum_{k,\alpha}(V_k)_{\beta\alpha}(V_k)^*_{\beta\alpha}\}) =$$

$$= W^*_{\alpha_0\beta}P_xW_{\alpha_0\beta} + \delta_{\alpha_0\beta}(-i[H_{\alpha_0},P_x] - \frac{1}{2}\{P_x,\sum_{\alpha\neq\alpha_0} W_{\alpha_0\alpha}W^*_{\alpha_0\alpha}\}) \tag{19}$$

Where the last equality holds owing to assumptions a) and b) above. Taking the $(\beta, i_\beta, j_\beta)$-th matrix element of (19) we see that it coincides with (18), thus, due to arbitrariness of $(\beta, i_\beta, j_\beta)$, we have proved Eq. (17).
Let $F$ denote the finite dimensional space of functions generated by $x \rightarrow <\psi|P_x|\phi>$. It is clear that $F = \{f : f(x) = Tr(AP_x),\ A \in \mathcal{A}_T\}$. So $dim\, F = dim\, \mathcal{A}_T$. We show that $F$ is the null space for $ker\, \pi$. Let $f(x) = \sum_{i,j} <\psi_i|P_x|\psi_j>$ and let $\mu_0 \in ker\, \pi$. Then

$$\mu_0(f) = \int \mu_0(dx) f(x) = \sum_{i,j} <\psi_i| \int \mu_0(dx) P_x |\psi_j> = 0$$

Moreover because $(A <\psi_i|P|\psi_j>)(x) = <\psi_i|L(P_x)|\psi_j>$ we have that $A : F \rightarrow F$ and so $S_t : F \rightarrow F$. It implies that $U_t : ker\, \pi \rightarrow ker\, \pi$ since $U_t\mu(f) = \mu(S_t f)$. Let $\hat{U}_t$ be the quotient semigroup. Then

$$\lim_{t\rightarrow 0} \frac{1}{t}[\hat{U}_t(P_x) - P_x] = \lim_{t\rightarrow 0} \frac{1}{t}[\pi(U_t\delta_x) - P_x] =$$

$$\lim_{t\rightarrow 0}( \int\int P(t,\, z,\, dy)\delta_x(dz)P_y - P_x) = (AP)(x),$$

so $\hat{U}_t$ and $T_t$ have the same generator and thus coincide. By Prop. 1 $U_t$ is associated with $T_t$. $\Box$

We can pass to the uniqueness problem. Let us consider a class of Markov processes associated with a general nonsymmetric Dirichlet form $\mathcal{E}$ on $L^2(E, dm)$ (here $dm|_{CP_\alpha}$ is a positive $U(\mathcal{H}_\alpha)$-invariant measure on Borel sets on $\mathbf{C}P_\alpha$) given by the closure of:

$$\mathcal{E}(u,v) = \int_E T(du, dv)\, dm + \int_E u(X.v)\, dm + \int_E (Y.u)v\, dm + \int_E uvc\, dm +$$

$$+ \int_{E\times E\setminus\Delta} (u(x) - u(y))\,(v(x) - v(y))\, J(dx, dy) \qquad (20)$$

for $u, v \in C^\infty(E)$. In (20) we have :

- $X, Y$ – smooth vector fields on $E$
- $c \in C^\infty(E)$

- $T$ – smooth (2,0)-tensor field, positively defined: $T(du, du) \geq 0$ for any $u \in C^\infty(E)$
- $\Delta = \{(x, x) \in E \times E\}$ – the diagonal
- $J(dx, dy)$ – positive symmetric Radon measure on $E \times E \setminus \Delta$ satisfying:
  - $\int_{E\times E\setminus\Delta} (u(x) - u(y))^2 \, J(dx, dy) < \infty$ for any $u \in C^\infty(E)$
  - the Radon derivative $\frac{J(dx,dy)}{dm(x)}$ exists and is a Borel measure
- for any $u \geq 0$ hold: $\int_E (cu + X.u) \, dm, \; \int_E (cu + Y.u) \, dm \geq 0$

It is worth to emphasize that such Dirichlet forms contain jumps, deterministic flows and diffusion processes as well. A straightforward calculation leads to the following result:

**Theorem 5.** The generator $B$ of the Dirichlet form $\mathcal{E}$ defined by (20) is given, in some coordinate system, by the formula

$$(Bu)(x) = (fu)(x) + \sum_i V^i(x)(\partial_i u)(x) + \sum_{ij} T^{ij}(x)(\partial_i \partial_j u)(x) - \int_E \mu(x, dy) \, u(y) \tag{21}$$

where $f \in C^\infty(E)$, $V$ – smooth vector field on $E$, $T$ – smooth, positive (2,0)-tensor field and $\mu$ – family of Borel signed measures, $u \in C^\infty(E)$ (the domain of $B$ comes from closing $(B, C^\infty(E))$ ). The detailed form of $f$, $V$ and $\mu$ is given by:

$$f = \frac{1}{M} X.M + \partial_i X^i - c$$

$$V = \frac{1}{M} T^{ij}(\partial_j M)\partial_i + (\partial_j T^{ij})\partial_i + X - Y$$

$$\mu(x, dy) = 2[\delta_x(dy) \frac{\int_{E\setminus\{x\}} J(dx, d\overset{\downarrow}{z})}{dm(x)} - \frac{J(dx, dy)}{dm(x)}]$$

and

$$\mu(x, \{x\}) = 2[\frac{\int_{E\setminus\{x\}} J(dx, d\overset{\downarrow}{z})}{dm(x)}],$$

where we have used the following notation

- $M$ is a coordinate of the volume form: $dm(x) = M(x)dx^1 \wedge \ldots \wedge dx^{2k}$ for $x \in \mathbf{C}P^k$
- $\delta_x(\cdot)$ – a measure concentrated in $\{x\}$
- an arrow indicates the variable the integral is evaluated over

**Remark 1.** The generator $B$ may be written in a fully invariant way:

$$(Bu)(x) = -cu - Y.u + \frac{1}{dm}\, L_{[T(du)+uX]}\, dm$$

where $L_Z\,\omega$ means the Lie derivative of the form $\omega$, associated to the vector field $Z$.

**Remark 2.** The proof of the Theorem 5 is straightforward – one should use the Stokes theorem for compact oriented manifold for the form

$$\alpha = v\, i_{[T(du)+uX]}\, dm$$

and because $E$ is without boundary ($\partial E = \emptyset$) we have $\int_E d\alpha = 0$. Evaluating $d\alpha$ we obtain the form of $B$.

Using Theorem 5 and the property that Tr $[L(P_x)] = 0\ \forall x \in E$ we conclude that $B(\mathbf{1}) \equiv 0$, $\mathbf{1}$-denotes the constant function taking value 1, and so $B$ can be written in the following form:

$$\begin{aligned}(Bu)(x) \ &= \sum_{ij} T^{ij}(x)(\partial_i\partial_j u)(x) \ + \sum_i V^i(x)(\partial_i u)(x) + \\ &\int_E \mu_0(x, dy)u(y) \ - \ \mu_0(x, E)u(x), \qquad (22)\end{aligned}$$

where $(T^{ij}(x))$ form a positive matrix and $\mu_0(x, dy)$ is a positive measure such that $\mu_0(x, \{x\}) = 0$ for every $x \in E$. Its domain $D(B)$ consists of $C^2$-functions.

**Lemma 6.** $(V_k)_{\alpha\alpha} = 0 \Rightarrow \forall \alpha \in \{1, \ldots, m\}\ \forall x, y \in \mathbf{C}P_\alpha$ such that $P_x \perp P_y$ the equality $Tr[P_y L(P_x)] = 0$ is satisfied.

**Proof**. let $x, y \in \mathbf{C}P_\alpha$ and $P_x \perp P_y$. Then

$$Tr[P_y L(P_x)] \;=\; -iTr(P_y[H_\alpha,\; P_x]) \;+ \sum_k Tr[P_y(V_k^* P_x V_k)_{\alpha\alpha}] -$$

$$\frac{1}{2}\sum_k Tr[P_y\{P_x,\; (V_k V_k^*)_{\alpha\alpha}\}] \;=\; \sum_k Tr[P_y(V_k^* P_x V_k)_{\alpha\alpha}]$$

But

$$(V_k^* P_x V_k)_{\alpha\alpha} = \; (V_k)^*_{\alpha\alpha} P_x (V_k)_{\alpha\alpha} \;=\; 0$$

so the assertion follows. □

We are now in position to show that the diffusion part is necessarily zero.

**Theorem 7.** $T^{ij}(x) \equiv 0$ for every i, j.

**Proof.** Because

$$B[Tr(P_y P)](x) \;=\; Tr[P_y L(P_x)]$$

so, by the above lemma, for every $\alpha$ and every $x, y \in \mathbf{C}P_\alpha$ such that $P_y \perp P_x$ we have that $B[Tr(P_y P)](x) \;=\; 0$. Let us denote the function $z \to Tr(P_y P_z)$ by $f_y(z)$. Then

$$(Bf_y)(x) \;= \int_{\mathbf{C}P_\alpha} \mu_0(x, dz) f_y(z) \;+ \sum_{ij} T^{ij}(x)(\partial_i\partial_j f_y)(x) \;+ \sum_i V^i(x)(\partial_i f_y)(x)$$

It is clear that $f_y$ is a smooth function and possesses a minimum at point $x$. So $\sum_i V^i(x)(\partial_i f_y)(x) \;=\; 0$ and we arrive at

$$\int_{\mathbf{C}P_\alpha} \mu_0(x, dz) f_y(z) \;+ \sum_{ij} T^{ij}(x)(\partial_i\partial_j f_y)(x) \;=\; 0$$

But $(\partial_i\partial_j f_y(x))$ and $(T^{ij}(x))$ are positive matrices so, by Schur's lemma, $(T^{ij}(x)\partial_i\partial_j f_y(x))$ is also a positive matrix. It follows that

$$\sum_{ij} T^{ij}(x)\partial_i\partial_j f_y(x) \;=\; 0$$

Now let us introduce a chart at point $x$, let say, $x = [(1,0,...,0)]$, $(U_0, \phi_0)$ such that

$$U_0 \;=\; \{[(z_0, z_1, \ldots, z_{n-1})] : \; z_i \in \mathbf{C}, \; \sum_i |z_i|^2 = \; 1, \; z_0 \neq 0\}$$

$$\phi_0[(z_0, z_1, \ldots, z_{n-1})] \ = (\frac{z_1}{z_0}, \ldots, \frac{z_{n-1}}{z_0}) \ = \ (x_1, y_1, \ldots, x_{n-1}, y_{n-1}),$$

where $x_i \ = \ Re\frac{z_i}{z_0}$, $y_i \ = \ Im\frac{z_i}{z_0}$. Then $\phi_0(x) \ = \ \vec{0} \in \mathbf{R}^{2(n-1)}$. Let us choose $y$ = [(0,1,0,...,0)]. It is clear that $P_y \perp P_x$ and so

$$\sum_{i,j=1}^{n-1} [T^{ij}_{x,x}(x)\frac{\partial^2(f_y \circ \phi_0^{-1})}{\partial x_i \partial x_j}(\vec{0}) \ + \ 2T^{ij}_{x,y}(x)\frac{\partial^2(f_y \circ \phi_0^{-1})}{\partial x_i \partial y_j}(\vec{0}) +$$

$$T^{ij}_{y,y}(x)\frac{\partial^2(f_y \circ \phi_0^{-1})}{\partial y_i \partial y_j}(\vec{0})] \ = \ 0$$

But for every $j \geq 2$ we have

$$\frac{\partial^2(f_y \circ \phi_0^{-1})}{\partial x_j^2}(\vec{0}) \ =$$

$$\lim_{h \to \infty} \frac{1}{h}[\frac{\partial(f_y \circ \phi_0^{-1})}{\partial x_j}(0, \ldots, x_j = h, 0, \ldots, 0) \ - \ \frac{\partial(f_y \circ \phi_0^{-1})}{\partial x_j}(\vec{0})] \ = \ 0$$

In the same way we prove that for every $j \geq 2$

$$\frac{\partial^2(f_y \circ \phi_0^{-1})}{\partial y_j^2}(\vec{0}) \ = \ 0$$

By positivity of the matrix $D^2(f_y \circ \phi_0^{-1})(\vec{0})$ we obtain that

$$T^{11}_{x,x}(x)\frac{\partial^2(f_y \circ \phi_0^{-1})}{\partial x_1^2}(\vec{0}) + 2T^{11}_{x,y}(x)\frac{\partial^2(f_y \circ \phi_0^{-1})}{\partial x_1 \partial y_1}(\vec{0}) + T^{11}_{y,y}(x)\frac{\partial^2(f_y \circ \phi_0^{-1})}{\partial y_1^2}(\vec{0}) = 0$$

Let $\lambda$ be an embedding $\lambda : \mathbf{C}P^1 \to \mathbf{C}P_\alpha$ given by

$$\lambda[(z_0, z_1)] \ = \ [(z_0, z_1, 0, \ldots, 0)]$$

It is clear that $x \ = \ \lambda(\vec{n_0})$ and $y \ = \ \lambda(\vec{n})$ for some unique $\vec{n_0}, \vec{n} \in \mathbf{C}P^1 \ = \ S^2$. Let $\psi_0$ be a chart at $\vec{n_0}$ given by

$$\psi_0 : \ \mathbf{C}P^1 - \{\vec{n}\} \to \mathbf{C}, \quad \psi_0(\vec{m}) \ = \ p \circ \phi_0 \circ \lambda(\vec{m}),$$

where $p \ = \ \mathbf{C}^n \to \mathbf{C}$ is the projection onto the first coordinate. So we may write that

$$a^{11}(\vec{n_0})\frac{\partial^2(f_{\vec{n}} \circ \psi_0^{-1})}{\partial q_1^2}(\vec{0}) \ + \ 2a^{12}(\vec{n_0})\frac{\partial^2(f_{\vec{n}} \circ \psi_0^{-1})}{\partial q_1 \partial q_2}(\vec{0}) +$$

$$+a^{22}(\vec{n_0})\frac{\partial^2(f_{\vec{n}}\circ\psi_0^{-1})}{\partial q_2^2}(\vec{0}) = 0,$$

where $a^{11}(\vec{n_0}) = T^{11}_{x,x}(x)$, $a^{12}(\vec{n_0}) = T^{11}_{x,y}(x)$, $a^{22}(\vec{n_0}) = T^{11}_{y,y}(x)$ and $q_1(\vec{m}) = x_1(\lambda(\vec{m}))$, $q_2(\vec{m}) = y_1(\lambda(\vec{m}))$. Let us change the chart $\psi_0$ onto spherical coordinates $(\theta,\varphi)$, $0\le\theta\le\pi$, $0\le\varphi\le 2\pi$ in such a way that $\theta(\vec{n_0}) = \pi/2$, $\varphi(\vec{n_0}) = 0$ i.e. $\vec{n_0} = (1,0,0)$ and $\theta(\vec{n}) = \pi/2$, $\varphi(\vec{n}) = \pi$ i.e. $\vec{n} = (-1,0,0)$. Because

$$f_{\vec{n}}(\vec{m}) = Tr(P_{\vec{n}}P_{\vec{m}}) = \frac{1}{2}(1+ <\vec{n},\vec{m}>) = \frac{1}{2}(1 - \sin\theta\cos\varphi)$$

so

$$\frac{\partial^2 f_{\vec{n}}}{\partial\theta\partial\varphi}(\vec{n_0}) = 0, \quad \frac{\partial^2 f_{\vec{n}}}{\partial\theta^2}(\vec{n_0}) = \frac{\partial^2 f_{\vec{n}}}{\partial\varphi^2}(\vec{n_0}) = \frac{1}{2}$$

which implies that $\tilde{a}^{11}(\vec{n_0}) = \tilde{a}^{12}(\vec{n_0}) = \tilde{a}^{22}(\vec{n_0}) = 0$, where $\tilde{a}^{ij}$ are the coefficients in the chart $(\theta,\varphi)$. But it is equivalent to

$$T^{11}_{x,x}(x) = T^{11}_{x,y}(x) = T^{11}_{y,y}(x) = 0$$

Changing $y = [(0,1,0,\ldots,0)]$ into $y = [(0,0,1,0,\ldots,0)]$ we obtain that

$$T^{22}_{x,x}(x) = T^{22}_{x,y}(x) = T^{22}_{y,y}(x) = 0$$

and so on. Thus, by the positivity, $T^{ij}(x) = 0$ for every $j,k$. Because $x$ was arbitrary the assertion follows. $\Box$

From the above theorem we conclude that the generator $B$ has to be of the following form

$$Bu(x) = V(x)u + \int_E \mu_0(x,dy)u(y) - \mu_0(x,E)u(x)$$

with domain $D(B) = C^1(E)$ as $B$ is a closed operator. To proceed further we first need a lemma.

**Lemma 8.** Let $X$ be a tangent vector to $\mathbf{C}P_\alpha$ at point $P_x$. Then $P_x + X \ge 0 \Leftrightarrow X = 0$.

**Proof.** Because $X \in T_x\mathbf{C}P_\alpha$ so $P_xX + XP_x = X$. It implies that

$P_x X P_x = 0$ and $P_x^\perp X P_x^\perp = 0$, where $P_x^\perp = I - P_x$. It means that in a basis $P_x\mathcal{H} \oplus P_x^\perp\mathcal{H}$ $X$ is of the form $\begin{pmatrix} 0 & X^* \\ X & 0 \end{pmatrix}$. So $P_x + X$ is a positive matrix if and only if $X = 0$. □

**Theorem 9.** $B = A$.

**Proof.** Because $A$ and $B$ are generators of semigroups which are associated with $T_t$ so for every $x \in E$ we have that $[(B - A)P](x) = 0$. Let $x \in \mathbf{C}P_{\alpha_0}$. Then

$$V(x)P + \sum_{\alpha=1}^{m} \int_{\mathbf{C}P_\alpha} \mu_{0,\alpha}(x, dy)P_y - \mu_0(x, E)P_x -$$
$$\sum_{\alpha \neq \alpha_0} c_\alpha(x)P_{x_\alpha} + c(x)P_x - v(x)P = 0,$$

where $\mu_{0,\alpha}(x, dy)$ denotes the restriction of $\mu_0(x, dy)$ onto $\mathbf{C}P_\alpha$. It is an operator valued equation so it has to be satisfied for every $\alpha$ separately. So for any $\alpha \neq \alpha_0$ we get

$$\int_{\mathbf{C}P_\alpha} \mu_{0,\alpha}(x, dy)P_y = c_\alpha(x)P_{x_\alpha}$$

which implies that $\mu_{0,\alpha}(x, dy) = c_\alpha(x)\delta(x_\alpha)(dy)$. For $\alpha_0$ we have

$$\int_{\mathbf{C}P_{\alpha_0}} \mu_{0,\alpha_0}(x, dy)P_y - \mu_0(x, E)P_x + c(x)P_x + V(x) - v(x) = 0$$

Let us introduce $a(x) = c(x) - \mu_0(x, E)$ and $w(x) = V(x) - v(x)$. Then taking the trace of the above equation we obtain $a(x) \leq 0$. Let us assume that $a(x) < 0$. It implies that

$$\frac{1}{|a(x)|} \int_{\mathbf{C}P_{\alpha_0}} \mu_{0,\alpha_0}(x, dy)P_y = P_x - \frac{1}{|a(x)|} w(x)$$

The left hand side of the above equation gives a positive operator and $w(x) \in T_x\mathbf{C}P_{\alpha_0}$ so, by Lemma 8, $w(x) = 0$. Thus we arrive at the contradiction because $\mu_{0,\alpha_0}(x, \{x\}) = 0$. So $a(x) = 0$ and we obtain that

$$\int_{\mathbf{C}P_{\alpha 0}} \mu_{0,\alpha 0}(x,\, dy)P_y + w(x) = 0$$

Evaluating the trace we get that $\mu_{0,\alpha_0}(x, \mathbf{C}P_{\alpha_0}) = 0$. Because it is a positive measure so it vanishes on every Borel subset of $\mathbf{C}P_{\alpha_0}$. So $w(x) = 0$ too and hence $A = B$. □

Thus we have the uniqueness. In the proof above we used repeatedly the fact that our Hilbert spaces were finite dimensional. In an infinite dimensional case the problem is much harder and we have no rigorous result. Our intuition is shaped here only by the infinitesimal argument of Section 3.

# 5 Conclusions

We have seen that the special class of couplings between a classical and a quantum system leads to a unique piecewise deterministic process on pure states of the total system that after averaging recovers the original master Liouville equation for statistical states. Irreversibility of the master equation describing time evolution of ensembles is reflected by going from potential to actual in the course of quantum jumps that accompany classical events. That is all fine but a natural question arises: what is classical? There are several options possible when answering this question. First of all the theory may be considered as phenomenological - then we choose as classical this part of the measurement apparatus (or observer) whose quantum nature is simply irrelevant for the given problem. Second, we may think of superselection quantities [40, 41] as truly classical variables. Some of them may play an important role in the dynamics of the measurement process - this remains for a while just a hypothesis. It is to be noticed that Jibu et. al (cf. [42], especially the last section 'Quantum Measurement by Quantum Brain'puts forward a similar hypothesis in relation to the possible role of microtubules in the quantum dynamics of consciousness.
Finally, a careful reader certainly noticed that in the formalism of EEQT one never really needs $C$ to be a *classical* system. It can be a quantum system as well. What is important it is that the the Liouville evolution preservers the diagonal of $C$. Thus the end product of the decoherence program [43, 44, 45], can be directly fed into the EEQT event engine. The uniqueness result above

- will be immediately relevant also for this case.

**Acknowledgements**
The early version of this paper was written when one of us (A.J) was visiting RIMS, Kyoto U. Thanks are due to Prof. H. Araki and for his kind hospitality and to Japanese Ministry of Education for the extended support. The third named author (R.O) acknowledges support of the Polish KBN grant no 2P30205707. A.J. also thanks for the support of the A. von Humboldt Foundation. We owe to Prof. Ph. Blanchard many discussions, encouragement and hospitality at BiBoS.

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